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Viscoelastic Cylinders of Complex Cross Section under Axial Acceleration Loads

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A KNOWLEDGE of the stresses and deformations in solid propellant rocket motors due to axial acceleration loads is necessary for analysis of motor structural integrity. This note deals with the axial acceleration of propellant grains of infinite length whose internal perforations are not circular but have a number of axes of symmetry, such as the common types of star perforations.

Equations of Motion

The equations of motion for a viscoelastic solid may be shown to be of the form

$$\left(K + \frac{G}{3}\right)\left[\frac{\partial e(t)}{\partial x}\right] + G\,[\nabla^2 u(t)] + X = \gamma\,\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

with permutations on the coordinates x,y,z, the displacements u,v,w, and the body forces X,Y,Z. The functional notation

$$G[f(t)] = f(t)G(0) + \int_0^t f(t_1)\,\frac{d}{dt_1}\,[G(t - t_1)]dt_1 \qquad (2)$$

is used. Here $G(t)$ is the viscoelastic shear relaxation modulus, $K(t)$ is the viscoelastic bulk relaxation modulus, γ is the material density, t is time, ∇^2 is the Laplacian operator, and e is the dilatation. This formulation, with the exception of acceleration and body force terms, has been given by Elder.[1]

Under the restriction that u, v, and w are invariant with z, the first two of the equations indicated by Eq. (1) reduce to the usual plane strain equations of linear viscoelasticity which do not contain the displacement w. The third equation which is now uncoupled from the first two, is

$$G\left[\frac{\partial^2 w(t)}{\partial x^2} + \frac{\partial^2 w(t)}{\partial y^2}\right] + Z = \gamma\,\frac{\partial^2 w}{\partial t^2} \qquad (3)$$

Henceforth, consideration will be given to the solution of Eq. (3) with geometries limited to cylinders having generators parallel to the z axis and body forces consisting only of the weight of the body. It is evident that the weight per unit volume can be expressed as the product of the density γ and a pseudo acceleration, the gravitational constant, and can thus be included in the right side of Eq. (3). Consequently, the body force will no longer be explicitly considered.

Boundary conditions applicable to Eq. (3) may consist of the specification of the displacement or shear stress as prescribed functions of time. Typically, these may take the form

$$w(t) = f(t) \text{ on } B_1 \qquad (4)$$

and

$$\tau_{nz}(t) = h(t) \text{ on } B_2 \qquad (5)$$

where B_1 and B_2 are the boundaries of a hollow cylindrical body, and n denotes the outward normal to the surface.

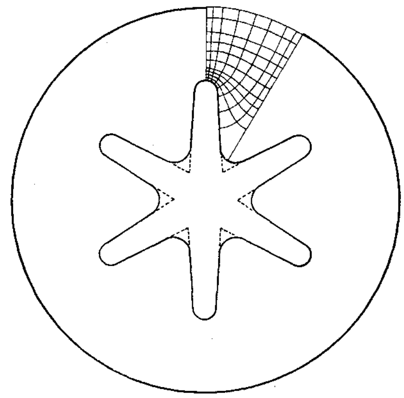

Fig. 1 Curvilinear co-ordinates obtained by conformal mapping.

The displacement of any point w can be considered to consist of two parts (w_1 and w_2), where w_1, a function of time only, is the displacement of boundary B_1 and w_2 is the displacement of the point relative to the boundary B_1. The displacement w_1 may be associated with a rigid body displacement. The displacement w_2 is associated with the deformation of the cylinder material and is a function of the space coordinates x and y and of time t. The displacement w_2 will be further restricted by neglecting dynamic effects so that the magnitude of the variation of w_2 with time is much less than that of w_1. Thus,

$$\partial^2 w/\partial t^2 \approx \partial^2 w_1/\partial t^2 \equiv A(t)$$

and since

$$\partial w_1/\partial x = \partial w_1/\partial y = 0$$

Eq. (3) may be written as

$$G\left[\frac{\partial^2 w_2(t)}{\partial x^2} + \frac{\partial^2 w_2(t)}{\partial y^2}\right] = \gamma A(t) \qquad (6)$$

where $A(t)$ is a specified acceleration $[A(t) = 0, t < 0]$. Specification of the over-all body acceleration may now be made independently of the displacements associated with deformations.

To obtain a solution of Eq. (6), it is convenient to use the Laplace transform of the equation with respect to time and to replace the transformed stress relaxation modulus $\bar{G}(s)$ in the resulting expression by the transformed creep compliance $\bar{J}(s)$ by using the relation

$$s\bar{G}(s) = [s\bar{J}(s)]^{-1}$$

Performing these operations and taking the inverse Laplace transform, results in

$$\nabla^2 L^{-1}\{\bar{w}_2(s)\} = L^{-1}\{\gamma s\bar{A}(s)\bar{J}(s)\} \qquad (7)$$

where $L^{-1}\{\ \}$ denotes the inverse Laplace transform. If a displacement function ψ is defined as

$$\psi = \frac{L^{-1}\{\bar{w}_2(s)\}}{L^{-1}\{\gamma s\bar{J}(s)\bar{A}(s)\}} \qquad (8)$$

Eq. (7) becomes

$$(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) = 1 \qquad (9)$$

which is valid regardless of the time dependence of the acceleration and regardless of the time dependence of the shear

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compliance. Furthermore, it can be seen from Eq. (9) that the displacement function ψ depends solely on the space coordinates x and y and on the boundary conditions.

If the acceleration $A(t)$ is applied as a step function, $A(t) = gH(t)$, where $H(t)$ is the unit step function, and g is a constant, then $\bar{A}(s) = g/s$, and Eq. (8) becomes

$$\psi = \frac{L^{-1}\{\bar{w}_2(s)\}}{L^{-1}\{g\gamma \bar{J}(s)\}} = \frac{w_2(t)}{g\gamma J(t)} \tag{10}$$

Thus, for a step function in the acceleration, the axial displacement at each point is directly proportional to the creep compliance.

Application to Star Geometries

The solution of Poisson's equation, Eq. (9), was considered for star geometries that are applicable to solid propellant grains. The internal perforation of such a propellant grain consists of p branches or star points, and the external boundary is circular. Wilson[2, 3] demonstrated the mapping of such regions by fitting the internal perforation with a mapping transform that maps the region exterior to the star shape in the $z^* = x + iy$ plane onto the region exterior to the unit circle in the $\zeta = \rho e^{i\theta}$ or transformed plane. Likewise, the circle defined by $\rho = \Gamma$ in the ζ plane corresponds to an irregular line in the z^* plane. However, if Γ is sufficiently large, the corresponding contour in the z^* plane is sufficiently regular to be considered circular.

The mapping function used by Wilson is

$$\omega(\zeta) = \sum_0^N C_n \zeta^{1-np} \tag{11}$$

Kantorovich and Krylov[4] show that under such a transformation, the Laplace operator transforms into an expression of the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{|\omega'(\zeta)|^2}\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{12}$$

Use of Eq. (12) allows Eq. (9) to be written as

$$\frac{\partial^2\psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial \theta^2} = |\omega'(\zeta)|^2 \tag{13}$$

After considerable manipulation of the mapping function $\omega(\zeta)$, it can be shown that $|\omega'(\zeta)|^2$ may be written as

$$|\omega'(\zeta)|^2 = \sum_{n=0}^{N} \frac{(1-np)^2 C_n^2}{\rho^{2np}} + 2\sum_{k=1}^{N}\left[\cos kp\theta \sum_{n=0}^{N-k} \frac{(1-np)(1-np-kp)C_nC_{n+k}}{\rho^{(2n+k)p}}\right] \tag{14}$$

The solution to Eq. (13) with the right side replaced by Eq. (14) is

$$\psi = \sum_{n=0}^{N} \frac{C_n^2}{4}\rho^{2(1-np)} + U\ln\rho + V + \sum_{k=1}^{N}\left\{H_k\rho^{kp} + L_k\rho^{-kp} + \sum_{n=0}^{N-k}\frac{C_nC_{n+k}}{2}\rho^{2-(2n+k)p}\right\}\cos kp\theta \tag{15}$$

This is the solution for the axial displacement function ψ in an infinite cylinder whose contour in the transverse cross section can be defined by a mapping function of the form indicated in Eq. (11).

Satisfaction of Boundary Conditions

In the second boundary condition, Eq. (5), where n is the outward normal, the shear stress can be expressed in terms of shear strain and, consequently, in terms of the axial displacement w_2 and normal displacement u_n. If the shear stress is taken to be zero, Eq. (5) may be written as

$$G[\partial w_2(t)/\partial n] = 0 \text{ on } B_2 \tag{16}$$

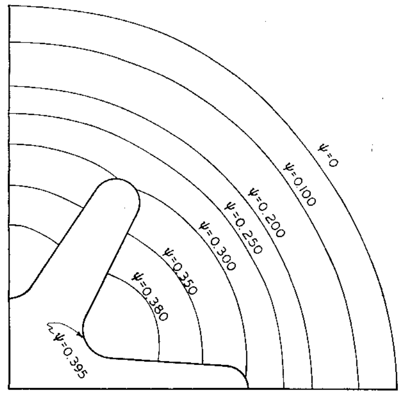

Fig. 2 Contour map of displacement function ψ.

Equation (16) is a homogeneous Volterra integral equation that has zero as the only continuous solution.[5] For the mapping function used, it can be shown that

$$\frac{\partial}{\partial n} = \pm \frac{1}{|\omega'(\zeta)|}\frac{\partial}{\partial \rho} \tag{17}$$

with the plus sign taken for an external boundary and the negative sign taken for an internal boundary.

If B_1 is defined as the outer boundary and B_2 is defined as the inner boundary, the boundary conditions in the transformed plane may be expressed as

$$w_2(t) = 0 \text{ on } B_1 \tag{18}$$

and

$$\partial w_2(t)/\partial \rho = 0 \text{ on } B_2 \tag{19}$$

Since these conditions are independent of time, they may be applied directly to the function ψ to yield

$$\psi = 0 \text{ on } B_1 \tag{20}$$

and

$$\partial \psi/\partial \rho = 0 \text{ on } B_2 \tag{21}$$

If $\rho = 1$ on B_1 and $\rho = \beta$ on B_2, the unknowns H_k, L_k, U, and V in Eq. (15) can be evaluated with the aid of Eqs. (20) and (21), which allows Eq. (15) to be written as

$$\psi = \sum_{n=0}^{N}\frac{C_n^2}{4}\left\{\rho^{2(1-np)} - \beta^{2(1-np)} - 2(1-np)\ln\left(\frac{\rho}{\beta}\right)\right\} + \sum_{k=1}^{N}\left\{\sum_{n=0}^{N-k}\frac{C_nC_{n+k}}{2}\left[\rho^{2(1-np)} - \beta^{2(1-np)} + \frac{[(2-np-kp)/k\beta + \beta^{2(1-np)}](\beta^{2kp} - \rho^{2kp})}{1+\beta^{2kp}}\right]\right\} \times \rho^{-kp}\cos kp\theta \tag{22}$$

Evaluation of Shear Stresses and Displacements

Since the displacements u and v in the z^* plane are zero for a loading of axial acceleration, the shear stress due to the axial acceleration load is

$$\tau_{lz}(t) = G[\partial w(t)/\partial l] \tag{23}$$

where l is any specified line lying in the z^* plane. By use of the Laplace transform, the shear stress can be written as

$$\tau_{lz}(t) = \gamma A(t)(\partial \psi/\partial l) \tag{24}$$

Usually, the maximum value of shear stress at a point is desired and may be obtained in the following manner. The relation of the derivative of ψ along a line l in the z^* plane to the derivative of ψ along a corresponding line λ in the ζ plane is given by

$$\frac{\partial \psi}{\partial l} = \frac{1}{|\omega'(\zeta)|}\frac{\partial \psi}{\partial \lambda} \tag{25}$$

In terms of the coordinates ρ and θ, this relation is

$$\frac{\partial\psi}{\partial\lambda} = \frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial\lambda} + \frac{\partial\psi}{\partial(\rho\theta)}\frac{\partial(\rho\theta)}{\partial\lambda} \tag{26}$$

The gradients of ρ and θ with respect to λ are known and given by the expressions $\partial\rho/\partial\lambda = \cos\alpha$, and $\partial(\rho\theta)/\partial\lambda = \sin\alpha$, where α is the angle between λ and the ρ coordinate line. When these relations are substituted into Eq. (26) and the resulting expression maximized with respect to α, there results the equation

$$\left.\frac{\partial\psi}{\partial\lambda}\right|_{\max} = \left[\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right]^{1/2} \tag{27}$$

which can be used with Eqs. (24, 25, and 27) to yield the maximum shear stress:

$$\tau_{\max} = \gamma A(t)\left[\frac{(\partial\psi/\partial\rho)^2 + (1/\rho^2)(\partial\psi/\partial\theta)^2}{|\omega'(\zeta)|^2}\right]^{1/2}$$

The displacement function ψ has been evaluated for the six-point, star-perforated shape shown in Fig. 1, which has been mapped with a 50-term mapping function. The original ballistic configuration is indicated by the dashed lines. The curvilinear coordinates are shown over a part of the cross section, demonstrating the slight mismatch between the true circular outer boundary and the curvilinear coordinate. A contour map of the displacement is shown in Fig. 2 for a step-function acceleration load, i.e., $A(t) = gH(t)$.

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Projectile Ranges in Excess of 180°

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Expressions are derived for the maximum range of a projectile where the range exceeds 180° in vacuum on a nonrotating spherical earth. The expressions are in terms of the burnout altitude and of the ratio of kinetic-to-potential energy at burnout. The corresponding angle between the burnout velocity vector and the local vertical is also determined. The results are presented graphically, and the limiting conditions are discussed.

I. Introduction

Blitzer and Wheelon[1] derived expressions for the maximum range of a projectile in vacuum on a nonrotating spherical earth and for the burnout angle corresponding to this range. The solution offered was in terms of a dimensionless parameter, the ratio of the kinetic to potential energy of the projectile at launch. This paper extends the meaning of maximum range to investigate trajectories in vacuum which result in impacts on a nonrotating spherical earth greater than 180° from the burnout point. Obviously, minimum energy criteria would rule out ballistic trajectories for ranges in excess of 180°. Such ranges, however, are of interest in range safety considerations for satellite launches and may be of some significance in ballistic missile flights where payload and accuracy would be sacrificed to avoid countermeasures along the shorter path. The method presented by Blitzer and Wheelon is used in this investigation.

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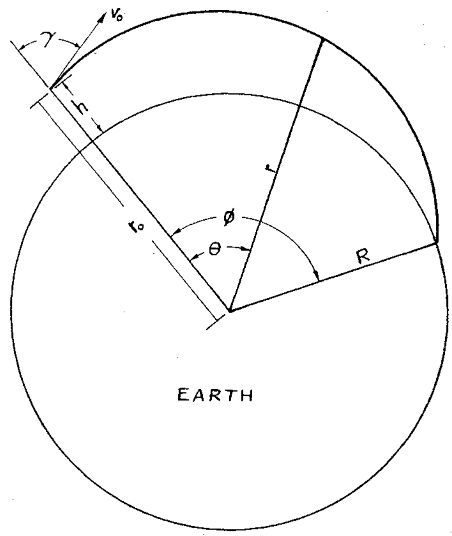

Fig. 1 Coordinate system.

The equations of motion for a particle moving in an inverse-square force field are[2]

$$\ddot{r} - r\dot{\theta}^2 + (GM/r^2) = 0 \tag{1}$$

$$(d/dt)(r^2\dot{\theta}) = 0 \tag{2}$$

where G is the constant of gravitation, and M is the mass of the earth. The initial conditions are (see Fig. 1):

$$\begin{aligned} r &= r_0 = R + h \qquad & \dot{r} &= \dot{r}_0 = v_0\cos\gamma \\ \theta &= \theta_0 = 0 & \dot{\theta} &= \dot{\theta}_0 = (v_0/r_0)\sin\gamma \end{aligned} \tag{3}$$

where v_0 is the burnout velocity, and γ is the burnout angle (the angle between the burnout velocity vector and the local vertical). A form of the solution, using Eqs. (1–3), is the familiar hit equation:

$$\frac{R+h}{R} = \frac{\sin(\gamma-\phi)}{\sin\gamma} + \frac{GM(1-\cos\phi)}{(R+h)v_0^2\sin^2\gamma} \tag{4}$$

where h is the burnout altitude, R is the radius of the earth, and ϕ is the (angular) range from the burnout point to the first intersection of the earth's surface by the projectile's orbit. The dimensionless parameter $\alpha \equiv (R+h)v_0^2/GM$ is twice the ratio of kinetic-to-potential energy at burnout. Substituting α in Eq. (4), one has

$$\frac{R+h}{R} = \frac{\sin(\gamma-\phi)}{\sin\gamma} + \frac{(1-\cos\phi)}{\alpha\sin^2\gamma} \tag{5}$$

II. Maximum Range Conditions

The minimum energy required to attain the range $\phi = 180°$ is expended for a given burnout altitude when $\gamma = 90°$ (orbit 1, Fig. 2). For the same burnout altitude h and holding $\gamma = 90°$, any increase in v_0 (increase in α) results in a new orbit with no intersection of the earth's surface (orbit 2, Fig. 2, e.g.). If the h and v_0 of orbit 2 are held constant